# Member's Default Utility Function for Default Fund Design Version 1 ("MDUF v1")

# Technical Paper No.3: Optimal Dynamic Strategies

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## 1 Introduction

This paper provides technical detail into the application of MDUF v1 to finding optimal retirement solutions. It should be read following a review of the "Introduction Paper", "Technical Paper No.1: MDUF v1 Design", and "Technical Paper No.2: Static Models".

As discussed in the "Introduction Paper", providing retirement outcome solutions is a hugely challenging and complex area. Technically the retirement outcomes problem can be defined as:

"A dynamic, integrated consumption and investment decision problem."

A dynamic solution accounts for both current market conditions and personal situation and is evaluated on a regular basis, e.g. every year. A static solution is a set of predetermined rules applied throughout retirement. Solution techniques required in a dynamic solution are more complex than those in a static solution.

In this paper we provide greater detail, in a more complex setting compared to the "Introduction Paper", around the application of MDUF v1 to finding the optimal dynamic retirement strategy that delivers the highest expected lifetime utility. For a comparison of static solutions, please refer to "Technical Paper No.2: Static Models".

The problem of finding the optimal retirement strategy using MDUF v1 has the following different dimensions and applications:

1. To seek optimal solution within a single product. For example, by recommending optimal consumption path and asset allocation.
2. To seek optimal solution across multiple products. For example, by recommending optimal mix of products.

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3. To quantify the value of advice by calculating the utility gain of the better retirement strategies.
4. To quantify the utility cost of suboptimal solution.
5. To help funds with project prioritisation.

# 2 Building Up the Framework in Detail

## 2.1 Key Variables

The utility function is designed in a way to jointly solve for optimal lifetime consumption patterns and allocation to risky assets. This can be represented by $O_t$ which denotes the vector that consists of choice variables at time $t$. A typical example is that individuals simultaneously make decisions around consumption and investment asset allocation, i.e.

$$O_t = (c_t,\ \omega_t) \quad 0 < t \leq T, \tag{2.1}$$

where $c_t$ is the annual consumption level at time $t$, and $\omega_t$ is the percentage weight of wealth invested in risky asset (equity) at time $t$.

Let $a$ denote the annuitisation ratio, which is a decision made at retirement. Let $b_t$ denote the wealth dynamics. The annuitisation ratio multiplied by the initial wealth is equal to the amount of wealth invested into life annuities. Let $L$ denote the price of a unit life annuity that pays \$1 p.a. throughout retirement. Let $l_t$ denote the annual amount of benefit from life annuities, we have

$$l_t = \frac{b_0 a}{L}. \tag{2.2}$$

The dynamics of wealth is

$$\begin{aligned} b_1 &= (b_0 - b_0 a + P_0 + l_0 - c_0)\left(1 + \tilde{R}_1\right), \\ b_{t+1} &= (b_t + P_t + l_t - c_t)\left(1 + \tilde{R}_{t+1}\right), \quad \text{for } t > 0 \end{aligned} \tag{2.3}$$

where $P_t$ is the amount of the Age Pension entitlement received at time $t$ if considered, and $\tilde{R}_t$ is the stochastic return p.a. of the investment portfolio from time $t-1$ to $t$.

## 2.2 Utility Function

$u(c_t)$ is the utility function defined over consumption at time $t$:

$$u(c_t) = \frac{c_t^{1-\rho}}{1-\rho}. \tag{2.4}$$

$v(b_t)$ is the utility function defined over end-of-life residual benefit:

$$v(b_t) = \frac{b_t^{1-\rho}}{1-\rho}\left(\frac{\phi}{1-\phi}\right)^{\rho}, \tag{2.5}$$

where $b_t$ is the level of wealth at time $t$ which equals the amount of residual benefit if the person dies between $t-1$ and $t$. $\rho > 0$ is the level of risk aversion; higher $\rho$ means greater risk aversion of the investors. $\phi$ is the strength of residual benefit motive; higher $\phi$ means stronger residual benefit motive.

## 2.3 Expected Lifetime Utility

The expected lifetime utility function (i.e. our MDUF v1), taking into account mortality, is therefore:

$$U_0 = \mathbb{E}_0\left[\sum_{t=0}^{T}\left\{{}_tp_x\ u(c_t) + {}_{t-1|}q_x\ v(b_t)\right\}\right], \tag{2.6}$$

where $x$ is the inception age of a particular cohort, $T$ is the retirement planning horizon ($x+T$ is the maximum age), $\mathbb{E}_0$ is the expectation operator with respect to time 0 (equivalently, age $x$), ${}_tp_x$ is the probability of being alive at age $x+t$ conditional on being alive at age $x$, and ${}_{t-1|}q_x$ is the probability of dying between age $x+t-1$ and $x+t$ conditional on being alive at age $x$.

## 2.4 Optimal Strategy: Utility Maximisation

The optimal strategy is a strategy that maximises the expected lifetime utility. The life-cycle utility maximisation problem is expressed as follows:

$$\max_{\{O_t,\ a\}_{0\le t\le T}} \mathbb{E}_0\left[\sum_{t=0}^{T}\left\{{}_tp_x\ u(c_t) + {}_{t-1|}q_x\ v(b_t)\right\}\right], \tag{2.7}$$

subject to

$$c_t \geq 0, \tag{2.8}$$

$$\omega_t \in [0,\ 1], \tag{2.9}$$

$$b_t \geq 0. \tag{2.10}$$

## 2.5 The Age Pension

### 2.5.1 Rules for account-based pension

The amount of the Age Pension entitlement is means-tested, i.e. depending on the current asset value and future incomes[1]. We formulate the following tests to determine the Age Pension entitlement:

- Asset test

$$P_t^A = \max\left(\bar{P} - \tau^A \max\left(b_t - \underline{b},\ 0\right),\ 0\right), \tag{2.11}$$

  where $P_t^A$ is the Age Pension entitlement under the asset test, $\underline{b}$ is the asset test threshold for full pension, $\bar{P}$ is the full Age Pension payment rate, and $\tau^A$ is the taper rate for asset test.

- Income test under deeming rule

$$P_t^I = \max\left(\bar{P} - \tau^I \max\left(r_1 \min\left(b_t,\ \underline{b}_1\right) + r_2 \max\left(b_t - \underline{b}_1,\ 0\right) - \underline{I},\ 0\right),\ 0\right), \tag{2.12}$$

  where $P_t^I$ is the Age Pension entitlement under the income test, $\underline{b}_1$ is the deeming threshold, below which a lower deeming rate $r_1$ is applied and above which a higher deeming rate $r_2$ is applied. $\underline{I}$ is the income test cut off point. $\tau^I$ is the taper rate for income test.

[1] See the Australia Government's website on the Age Pension at `https://www.humanservices.gov.au/customer/services/centrelink/age-pension`

- Combining the two tests

$$P_t = \min\left(P_t^A,\ P_t^I\right). \tag{2.13}$$

### 2.5.2 Rules for life annuities

Based on the Age Pension eligibility rules for lifetime income streams, we formulate the following tests to determine the Age Pension entitlement:

- Asset test

$$P_t^A = \max\left(\bar{P} - \tau^A \max\left(b_t^p + b_t^a - \underline{b},\ 0\right),\ 0\right), \tag{2.14}$$

  where

$$b_t^a = \max\left(b_0^a - \frac{b_0^a}{\mathring{e}_x} \times t,\ 0\right), \tag{2.15}$$

  $b_t^p$ and $b_t^a$ are the time $t$ value of assets in account-based pension and annuity respectively. $b_0^a$ is the purchase price of annuity at time 0. $\mathring{e}_x$ is the life expectancy of the member age $x$ when the annuity is purchased at time 0.

- Income test

$$P_t^I = \max\left(\bar{P} - \tau^I \max\left(r_1 \min\left(b_t^p,\ \underline{b}_1\right) + r_2 \max\left(b_t^p - \underline{b}_1,\ 0\right) + \max(I_t^a - \frac{b_0^a}{\mathring{e}_x},\ 0) - \underline{I},\ 0\right),\ 0\right), \tag{2.16}$$

  where $I_t^a$ is the annuity income at time $t$ and $\frac{b_0^a}{\mathring{e}_x}$ is the deduction amount under the Age Pension rule for annuity.

- Combining the two tests

$$P_t = \min\left(P_t^A,\ P_t^I\right). \tag{2.17}$$

# 3 Optimal Strategy Solution Technique

This section focuses on the complex computational techniques required in solving for optimal dynamic strategies. To fully understand the detail in this section, it would require knowledge in business school postgraduate courses and equivalent or higher levels. Readers who find the detailed solution technique too difficult can skip this section and proceed to the next section on case studies.

Using the Bellman equation, the utility maximisation problem in Equation (2.7) can be expressed in the following recursive equation:

$$V(b_t) = \max_{O_t,\ a}\left\{u(c_t) + \mathbb{E}_t\left[p_{x+t}\ V(b_{t+1}) + q_{x+t}\ v(b_{t+1})\right]\right\}, \tag{3.1}$$

where $V(b_t)$ denotes the maximised utility based on information up to time $t$, $p_{x+t}$ is the annual survival probability of an individual aged $x+t$, and $q_{x+t}$ is the annual death probability of an individual aged $x+t$. This recursive equation in essence ensures that the optimum of the optimum is the global optimum.

First-order conditions and envelop conditions are then derived based on the Bellman Equation (3.1). See the remaining part of this section for details on first-order condition and envelop condition. Optimal solution is directly coded based on the derived first-order condition. The envelop condition is useful in that it gives us the formula for an important component in the first-order condition.

The solution technique consists of two separate stages. The first stage is backward induction. In this stage, we start from the terminal period, define grid points of wealth dynamics, and calculate optimal choice using first-order conditions and envelop conditions. The second stage is forward simulation. Once the optimal solution for each grid point of wealth is obtained in the first stage, forward simulations are then used to demonstrate optimal paths of consumption and asset allocation. The resulting dynamics of wealth can then be calculated using these forward simulations and the corresponding optimal values in the choice variables.

# 4 Welfare Analysis

Utility can be used as a scoreboard of an individual's lifetime welfare, which is a ranking instead of an absolute score. It is difficult to perform quantitative comparisons directly based on utility scores.

There are indeed other measures related to utility which are widely used to investigate welfare gains or losses of different retirement strategies. These measures can also be used in our case to assess retirees' welfare gains of having access to the Age Pension provided by the Australian government and to life annuities. This section describes key measures related to utility. The next section will show numerical results based on our model set-up.

## 4.1 Metrics on Welfare Gains

### 4.1.1 Certainty Equivalent Consumption (CEC)

Certainty Equivalent Consumption is calculated as the consumption level in the one-period utility function, i.e. Equation (2.4), that equates the utility level to the optimal lifetime utility. CEC, in essence, is a monotonic transformation of the optimal lifetime utility. A higher level of optimal lifetime utility also corresponds to a higher CEC level. Note that CEC does not necessarily convey information of the actual level of consumption.

### 4.1.2 Wealth Gap (WG)

For a pair of two cases, the wealth gap is calculated as the additional amount in the initial wealth in one case that can result in the same level of optimal lifetime utility in the other case. When we compare Case 2 vs. Case 1 and Case 3 vs. Case 2, the wealth gap measures the dollar amount of welfare gains, reflected in the initial wealth level, of having access to the Age Pension and life annuities, respectively.

### 4.1.3 Extra Annual Return (EAR)

For a pair of two cases, the extra annual return is calculated as the additional annual return of the fund investment performance in one case that can result in the higher level of optimal lifetime utility in the other case. When we compare Case 2 vs. Case 1 and Case 3 vs. Case 2, the extra annual return measures welfare gains, reflected in terms of annual investment gains, of having access to the Age Pension and life annuities, respectively.

# 5 Case Studies

Our case studies are centred around the account-based pension (ABP) for non-homeowner single males. The reason for focusing on non-homeowners is that taking into account housing assets would require a good analysis of the housing asset dynamics and would complicate our illustrations. Our case studies are for singles instead of couples, as incorporating joint mortality and spouse information would further complicate the model.

Two financial assets and two retirement products are included in our case studies. The two financial assets are a risk-free asset and a risky asset (equity). We assume that the real risk-free rate is fixed at $r_f$ p.a. for the horizon considered. Real returns for risky assets are assumed to follow an identical and independent normal distribution, i.e.:

$$R_t \sim N(\mu_R,\ \sigma_R^2).$$

The two additional retirement products include the Age Pension and life annuities. The Age Pension entitlements are means-tested. The rules are detailed in Section 5.2. The features and pricing methods of life annuities are described in Section 5.3.

We show numerical results for the above proposed life-cycle model with and without taking into account the Age Pension and life annuities.

Mortality rates are sourced from the Australian Life Tables 2010-12 by Australian Government Actuary[2]. Survival curves and death distributions for males and females are shown in Figure 5.1.

For illustration purposes, we only show case study results for males in this paper. The results for females can be easily replicated by using the corresponding female mortality rates.

## 5.1 Case 1: Base Case

We start with a scenario where the Age Pension is not taken into account and life annuities are not available. Parameter values are shown in Table 5.1.

Table 5.1: Parameter values for base-case analysis. Sources are cited in brackets.

| **Parameter** | **Explanation** | **Value** | **Source** |
|---|---|---|---|
| $r_f$ | Risk-free rate | 0.00% | Assumption |
| $\mu_R$ | Mean equity return | 5.00% | Assumption |
| $\sigma_R$ | Standard Deviation of equity return | 15.00% | Assumption |
| $\rho$ | Risk aversion | 8 | MDUF v1 Specification |
| $\phi$ | Residual benefit motive strength | 0.83 | Lockwood (2014) |
| $b_0$ | Initial wealth ($1,000) | 500 | Assumption |
| $P_t$ | The Age Pension Entitlement ($1,000) | 0 | Assumption |

[2]The Australian Life Tables 2010-12 can be downloaded via `http://www.aga.gov.au/publications/life_table_2010-12/`

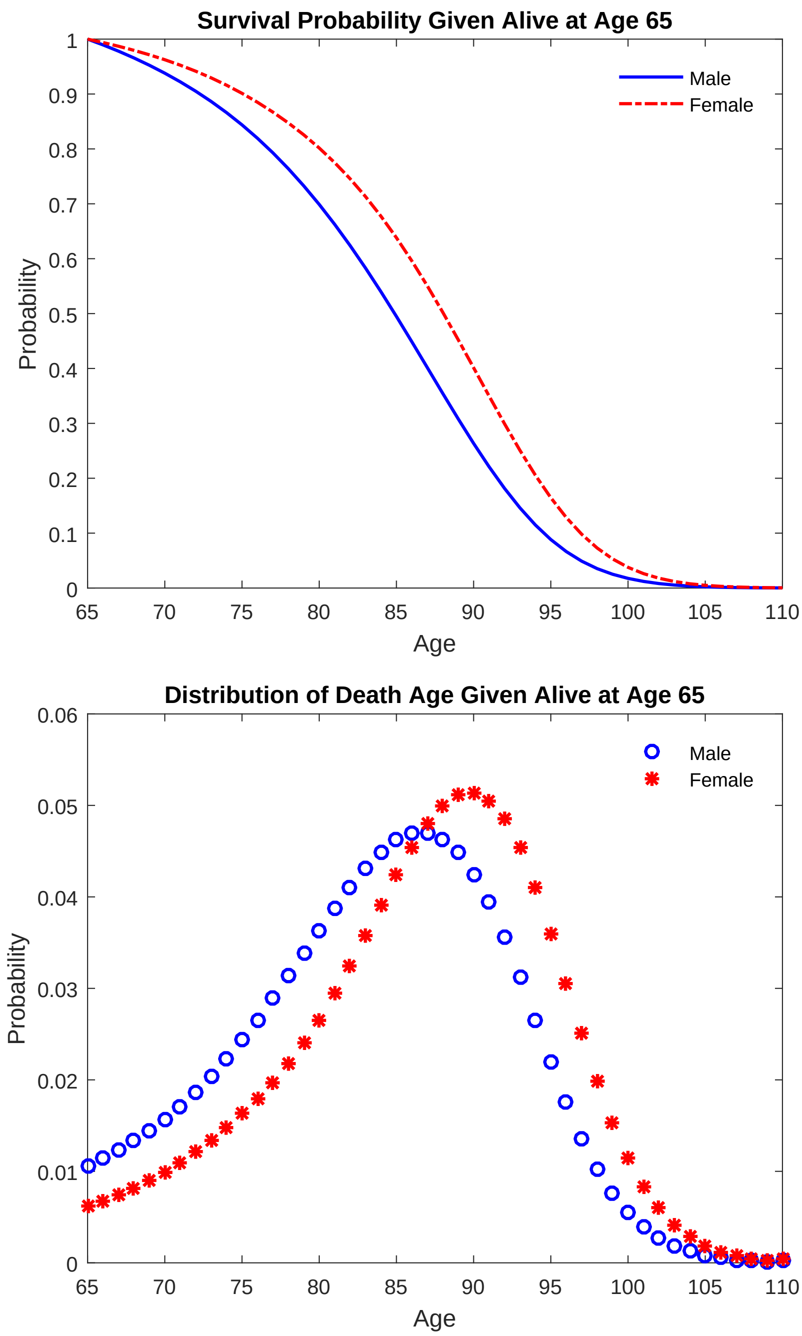


Figure 5.1: Survival curves and death distributions conditional on surviving to age 65.

We note that the values for $\rho$ and $\phi$ in Table 5.1 are the recommended values under MDUF v1. We observe that a risk aversion parameter of 8 delivers fairly reasonable variability in year-to-year consumption changes. This level is also within the range (1 to 10) used in many academic studies (for instance, Ameriks et al., 2011; Yogo, 2009; Mehra and Prescott, 1985; Friend and Blume, 1975). Please see details in "Technical Paper No.1: MDUF v1 Design".

The optimal average consumption paths, asset allocation, wealth paths, and propensity to consume (consumption as a proportion of wealth) are shown in Figure 5.2.

We can see from Figure 5.2 that the optimal allocation to risky asset is a constant proportion of wealth, which is 33.95%. This constant proportion is due to the CRRA specification of utility function. See "Technical Paper No.1: MDUF v1 Design" for explanation. The optimal

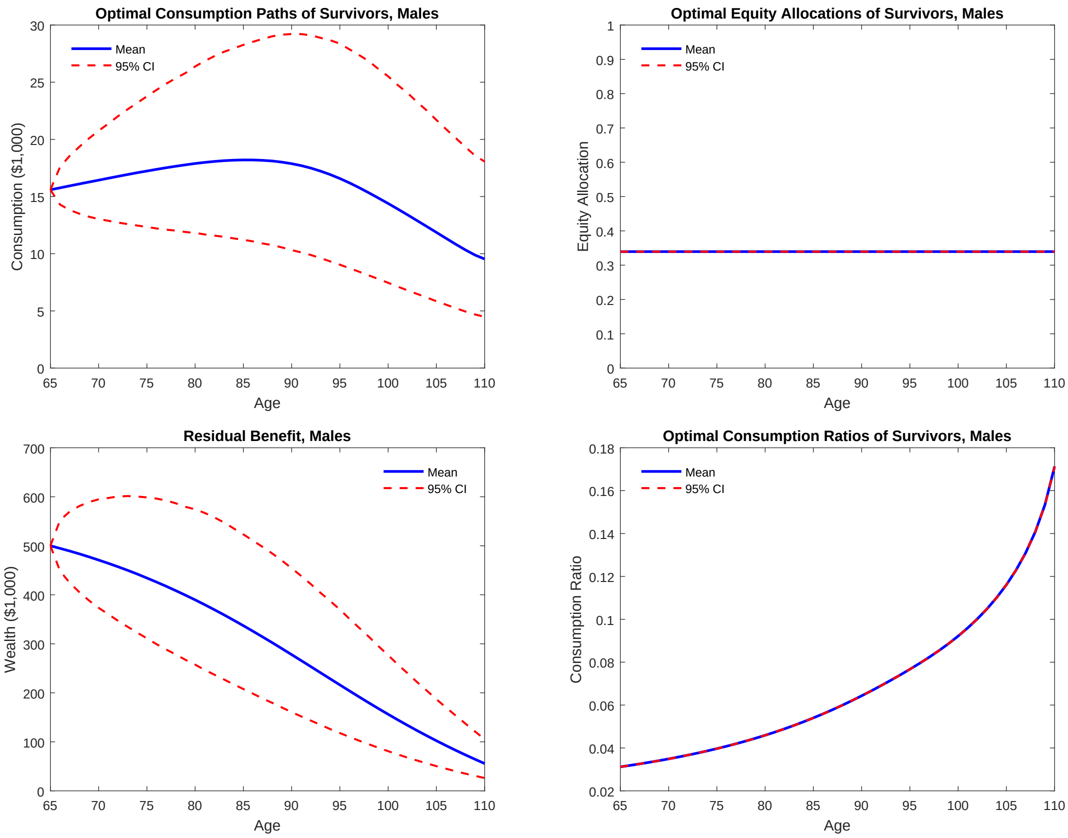


Figure 5.2: Optimal average consumption, asset allocation and wealth paths.

average consumption path slightly increases for the first 20 years. This represents retirees sacrificing their consumption in the first few years for potentially more accumulation in their wealth, which is also motivated by their desire to hedge the risk of living longer than expected. The average consumption path starts to decrease from age 90 onward, which is largely due to the consideration of (idiosyncratic) longevity risk, or survival probability scaling. The average wealth path shows a generally decreasing pattern, which is a mixed result of risky asset growth and wealth draw-down. The red dashed lines indicate the 95% confidence intervals.[3] Variations are caused by investment risk and idiosyncratic mortality risk.

Another observation is the very stable (as shown by the almost deterministically increasing pattern) propensity to consume, which is calculated as the ratio of consumption to wealth. The consumption ratio starts from 3%-4% (similar to the traditional rule of thumb Bengen (2004)) at retirement and more than doubles when the retiree reaches age 88. At the maximum attainable age (i.e. 110 in our study), the propensity to consume is 17%, which is equal to 1 minus the residual benefit motive strength ($\phi$). This relationship is not a coincidence, but indeed is due to the definition of residual benefit motive strength. The increasing pattern is very comparable to the ABP minimum drawdown rules, under which the consumption ratio is 4% at age 65 and increases to 14% at age range 95-110.

## 5.2 Case 2: The Impact of the Age Pension

In this section, we incorporate the Age Pension and follow the Australian Government's eligibility rules, as formulated in Section 2.5. Based on the information provided on the website, the

[3] 95% confidence intervals are the bounds within which possible outcomes would fall with a probability of 0.95.

parameter values in the formulae in Section 2.5 are shown in Table 5.2.

Table 5.2: The Age Pension eligibility and payment rates as at 1 July 2016. Single and non-home owner rates are used. Information is sourced from the Australian Government Department of Human Service website. Note that Effective from 1 January 2017, there are a few changes to these parameter values. The results in this technical paper do not incorporate these updates.

| Parameter | Explanation | Value |
|---|---|---|
| $\bar{P}$ | The Full Age Pension payment rate (p.a.) | \$22,721.4 |
| $\tau^A$ | Taper rate under the asset test | 0.0015 |
| $\underline{b}$ | Threshold for full pension under asset test | \$360,500 |
| $\underline{b}_1$ | Threshold for different deeming rates under income test | \$49,200 |
| $r_1$ | Lower deeming rate | 1.75% |
| $r_2$ | Higher deeming rate | 3.25% |
| $\underline{I}$ | Income test cut off point (p.a.) | \$4,264 |
| $\tau^I$ | Taper rate under the income test | 0.5 |

The optimal average consumption paths, asset allocation, wealth paths, and propensity to consume (consumption as a proportion of wealth) are shown in Figure 5.3.

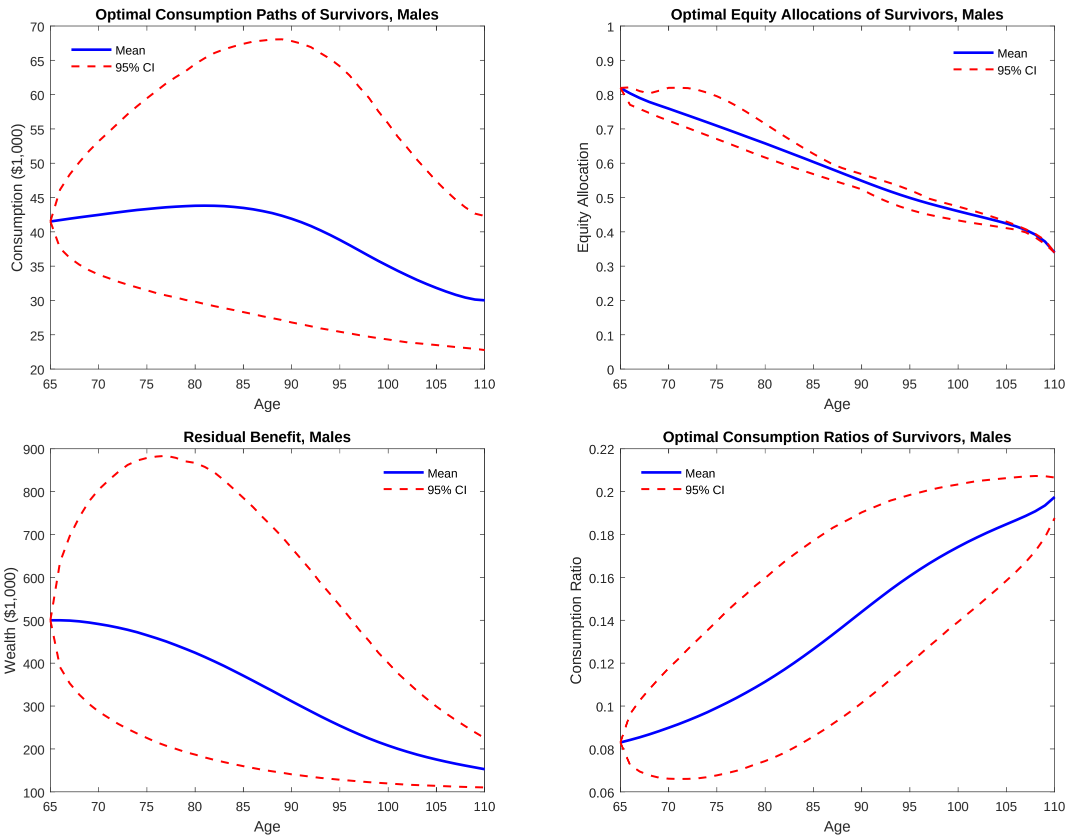


Figure 5.3: Optimal average consumption, asset allocation and wealth paths when the Age Pension is taken into account.

Compared with Case 1, we see a generally higher proportion of wealth that should be invested in equity, especially in early years. The optimal allocation to equity is 80% and shows an almost linearly decreasing trend as the retiree ages. This decreasing pattern with respect to age is different from the base case results, but as the individual survives to an old age the allocation to equity converges to the constant proportion in Case 1 (i.e. 33.95%). This is because in the terminal period there is no future Age Pension, resulting in the same problem as in Case 1. Therefore the optimal asset allocations in the two cases are the same for the terminal period. The greater the distance from the terminal period, the higher the Age Pension entitlement is expected to be, resulting in larger differences in the asset allocations in the two cases.

The dollar amount of consumption is higher than the general consumption level in the base case. The bell shape of average consumption is also a feature of this case study. See more comparative analysis of this in Section 5.4. As expected, the consumption ratio is significantly higher than the base case where there is no Age Pension. We also see some variation around consumption ratios when the Age Pension is taken into account. The reason is that the optimal consumption not only depends on the current wealth level but also on another source - the Age Pension.

The wealth level is indeed generally higher than that in Case 1, due to the Age Pension supplements. The average wealth level almost decreases to around \$150k in the terminal period. For a comparison, the average wealth in Case 1 decreases to around \$50k in the terminal period.

The amount of the Age Pension received is shown in Figure 5.4. The figure shows the average and 95% confidence interval of the Age Pension entitlements. The average amount of the Age Pension received slightly decreases for the first five years and starts to steadily increase onward. The slight decreasing pattern for the first five years is largely due to the non-linear relationship between the Age Pension entitlement and wealth level. After the first five years, the average value of the Age Pension entitlement starts to increase because the level of wealth is likely reduced due to draw-down so the amount of the Age Pension entitlements becomes high.


Figure 5.4: The Age Pension Entitlements in Case 2.

To summarise, the Age Pension functions as a safety net for retirees. It increases the general level of consumption and motivates retirees to invest more wealth in risky assets. The welfare improvement from having access to the Age Pension is analysed in Section 4.

## 5.3 Case 3: The Role of Life Annuities

In this case, we investigate the role of life annuities in retirees' retirement planning, in particular focusing on consumption smoothing and longevity risk protection.

We use life annuities that provide inflation-linked annual benefits. For illustration purposes, we assume life annuities are priced using the risk-free interest rate. Other forms of loadings, such as expense loading, profit loading, and the cost of capital requirements, are not explicitly included in the pricing.

Note that this case study aims to provide an illustration of the impact of life annuities on retiree's lifecycle strategy, so the numerical results should not be used in any way for specific financial advice.

The annuitisation ratio, calculated as the ratio of annuitised wealth to the initial wealth, is one of the control variables as stated in Section 2. We find the annuitisation ratio along with consumption and asset allocations that would generate the highest expected lifetime utility, as formulated in Section 2.

Figure 5.5 shows the level of Certainty Equivalent Consumption (CEC) for different annuitisation ratios. The relationship shows a bell shape. As shown in the figure, the optimal proportion of wealth that should be annuitised is around 45% for retirees with half a million dollars in their super fund at retirement. Annuitising more than 80% of the initial wealth would result in a utility level even worse than without any annuities. Key reasons include the residual benefit motive, the crowd-out effect of the Age Pension, and the assumed low interest rate in annuity pricing. Annuitising too much wealth will result in a very low value in the residual benefit, which places a large penalty on the lifetime utility if the residual benefit motive is relatively high. The Age Pension also provides longevity protection, so it can have a substituting impact on the demand for life annuities. In our case a low interest rate is assumed in the pricing, which means the price of life annuity is expensive, so the demand for life annuities is also reduced.

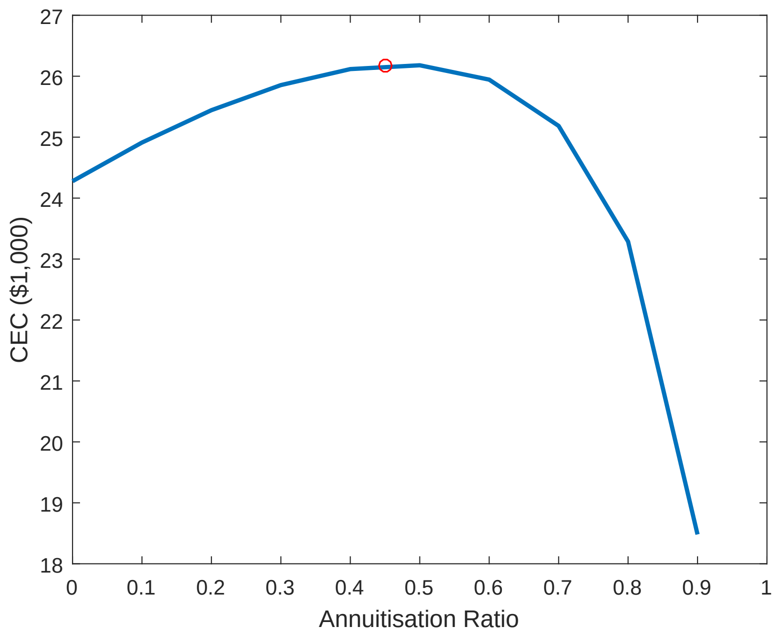


Figure 5.5: Certainty Equivalent Consumption (CEC) for different annuitisation ratios. The highest CEC is marked in red circle.

The optimal average consumption paths, asset allocation, wealth paths, and propensity to consume (consumption as a proportion of non-annuitised wealth) are shown in Figure 5.6. We see that the optimal expected consumption path becomes flatter than that in Case 1 and Case 2. This reflects the consumption smoothing function of purchasing life annuities. The average annual consumption at very old ages is above $40,000 p.a. and the lower 95% confidence interval of annual consumption at very old ages is around $33,000. By comparison, the average consumption at very old ages in Case 2 is above $30,000 and the lower 95% confidence interval is around $25,000. This shows the longevity risk protection provided by life annuities.

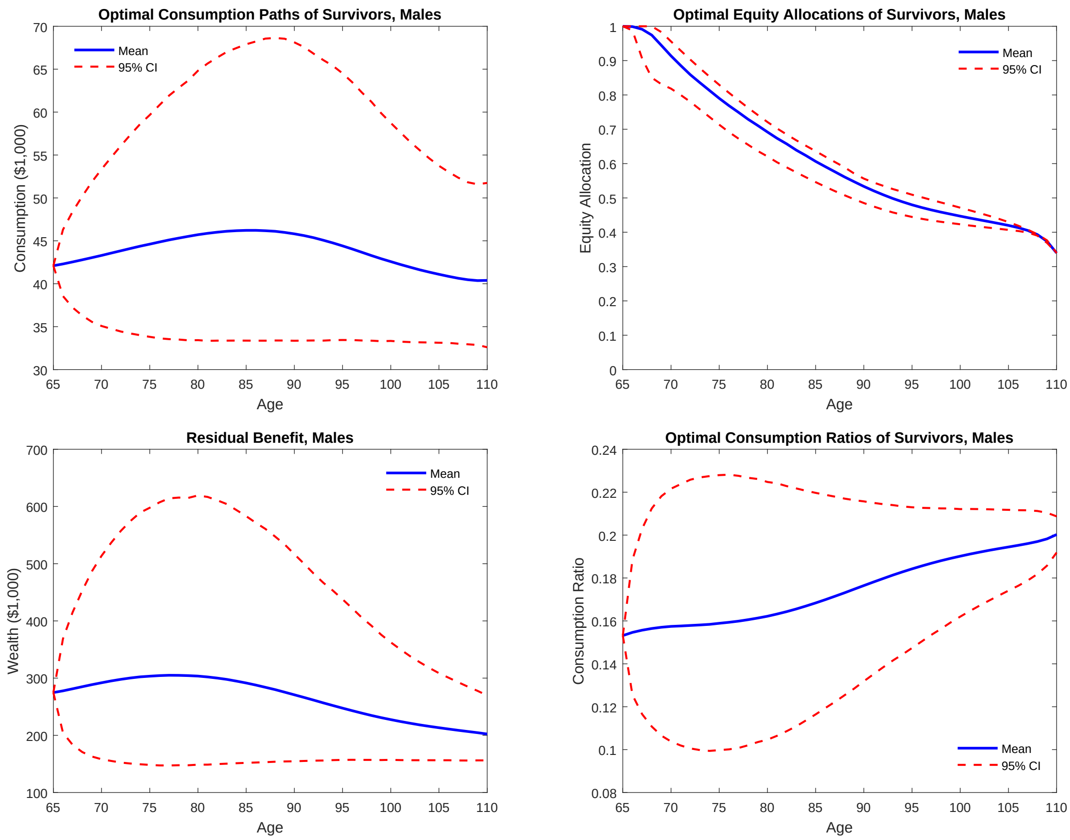


Figure 5.6: Optimal consumption, asset allocation and wealth paths when the Age Pension and life annuities are taken into account. Note that equity allocation and consumption ratio are quoted as the proportion of non-annuitised wealth.

We can see that the expected consumption ratio as a proportion of non-annuitised wealth for early retirement in Case 3 almost doubles that in Case 2. The key reason is higher consumption level as well as lower dollar amount of non-annuitised wealth, which can be confirmed by the residual benefit in the bottom left panel of Figure 5.6.

The equity allocation as a proportion of non-annuitised assets in Case 3 is generally higher than that in Case 2. It starts at 100% and decreases all the way down to 33.95%. Comparisons of asset allocations for the three cases are discussed in Section 5.4.3.

The Age Pension entitlements at different ages are shown in Figure 5.7. The figure shows the average and 95% confidence intervals of the Age Pension entitlements. The average amount of the Age Pension received shows a similar increasing pattern in general as in Case 2, except that in Case 2 the average of the Age Pension entitlement decreases for the first few years. A detailed comparison of the Age Pension entitlements between the two cases is shown in Section 5.4.4.

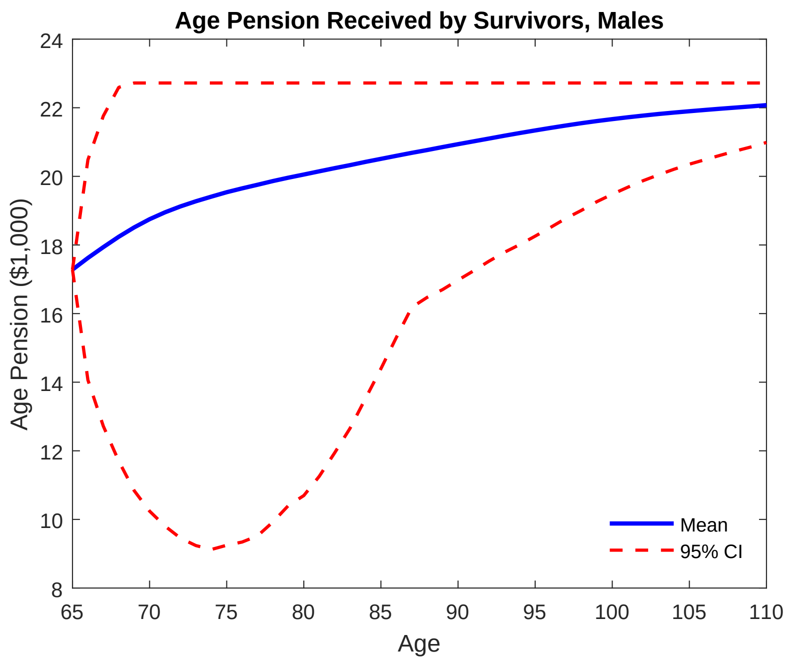


Figure 5.7: The Age Pension Entitlements in Case 3 (with access to life annuities).

## 5.4 Comparative Analysis

This section provides a comparative analysis across the three cases.

### 5.4.1 Consumption Comparison

The Association of Superannuation Funds of Australia (ASFA) provides retirement standards at the level of living a modest lifestyle and of living a comfortable lifestyle, respectively. As of 2016 for 65-year-old homeowner singles, the modest lifestyle standard is $23,797 p.a. and the comfortable lifestyle standard is $43,184.[4] We compare our expected consumption levels in the above three cases with ASFA retirement standards. Note that the ASFA standards are for homeowners whereas our case studies focus on non-homeowners. We therefore encourage caution in directly comparing the results in our case studies with ASFA standards. The results are shown in Figure 5.8.

As indicated by the black dotted line in Figure 5.8, for a 65-year-old male with half a million dollars retirement savings, the expected consumption level is below the ASFA modest standard for most of the time in Case 1, i.e. if the retiree has no access to the Age Pension or life annuities. When the Age Pension is accessible (Case 2), the expected consumption level is substantially higher than the ASFA Modest level and reaches the level around the ASFA comfortable standard in the first 20 years and starts to decrease for older ages, as indicated by the red dashed line. This shows that the Age Pension can provide the retiree with streamlined retirement incomes to improve the living standard. Annuitising a proportion of retirement assets can result in a higher level of consumption, especially for old ages. Due to the longevity protection provided by life annuities, the consumption levels for older ages in Case 3 (blue solid line) are higher than those in Case 2 (red dashed line).

[4]Data are obtained from ASFA's website via `http://www.superannuation.asn.au/resources/retirement-standard`, on 24 May 2016.

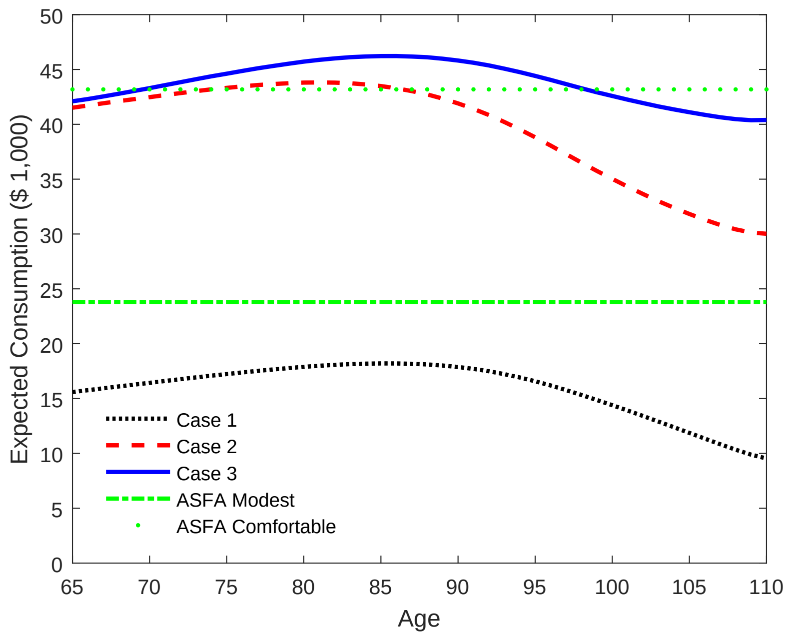


Figure 5.8: Comparison of expected consumption levels for different cases and ASFA retirement standards. In Case 1, retirees have no access to the Age Pension or to life annuities; In Case 2, retirees have access to the Age Pension; In Case 3, retirees have access to the Age Pension as well as life annuities.

### 5.4.2 Residual Benefit Comparison

Figure 5.9 compares the expected residual benefit at death for the three cases. Comparing Case 1 and Case 2, we observe higher residual benefit when the Age Pension is accessible, which stems from the positive residual benefit motive. When purchasing life annuities, the residual benefit is lower for earlier ages, but becomes higher for those who survive to very old ages, i.e. after age 97. Note that the probability of a 65-year-old Australian male surviving to age 97 is less than 5%. Taking into account survival probabilities, we can observe that life annuities increase the lifetime consumption most likely at the cost of a reduction in the residual bnefit.

### 5.4.3 Asset Allocation Comparison

Asset allocations are compared across the three cases. The results are shown in Figure 5.10. We observe substantially higher allocations to risky assets if the Age Pension is taken into account. This is largely due to the income protection provided by the Age Pension. When life annuities are accessible the optimal allocation to equities as a proportion of non-annuitised wealth is higher than that in Case 2 for earlier ages, but this discrepancy becomes very small after Age 80. Due to the fact that the optimal strategy is to annuitise around 45% wealth at retirement, the optimal dollar amount of equity investment in Case 3 is indeed lower than that in Case 2. The bottom panel of Figure 5.10 shows the dollar amount of allocations to different assets at retirement. We observe lower allocations to risky asset in Case 3 than in Case 2.

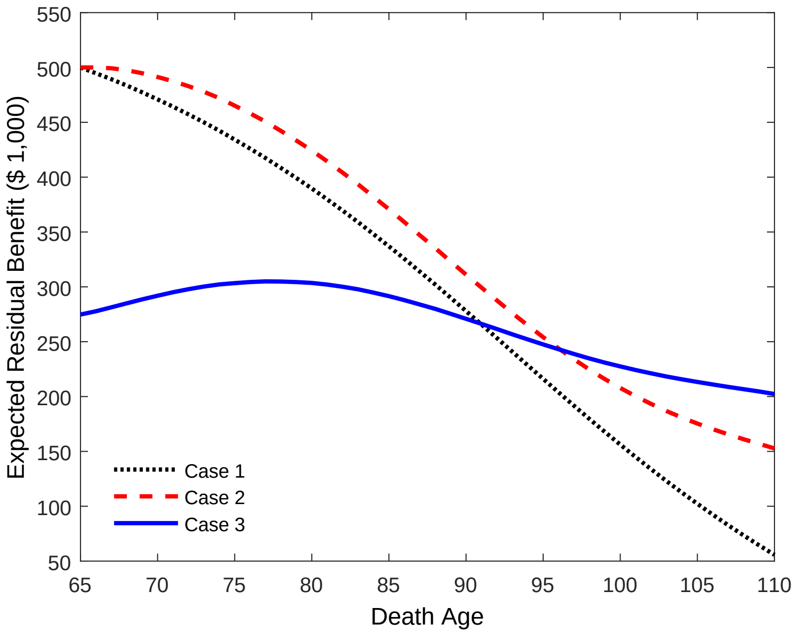


Figure 5.9: Comparison of expected residual benefit at death. In Case 1, retirees have no access to the Age Pension or to life annuities; In Case 2, retirees have access to the Age Pension; In Case 3, retirees have access to the Age Pension as well as life annuities.

### 5.4.4 The Age Pension Entitlement Comparison

The amount of the Age Pension entitlement is compared between Case 2 and Case 3. The results are shown in Figure 5.11. We can see the advantages of life annuities in terms of the Age Pension entitlements. The advantages are twofold: the average amount of the Age Pension is higher for earlier ages (e.g. before age 98 in our case studies); downside variations are lower except for very old ages (i.e. after age 105). The enhancement in the Age Pension entitlement for annuities stems from each of the two tests: under the income test, part of annuity benefit is not considered as income but as the return of capital; under the asset test, annuity asset is assumed to depreciate linearly until reaching the life expectancy.

## 5.5 Welfare Analysis Results

Tables 5.3 and 5.4 show the optimal lifetime utility and the three metrics used for assessing welfare gains of having access to the Age Pension and life annuities. Note that the results are for retirees with initial wealth of $500k. The results would vary for different wealth levels.

Table 5.3: Value functions and CECs of the three cases.

| | **Case 1** | **Case 2** | **Case 3** |
|---|---|---|---|
| **Optimal Lifetime Utility** ($10^{-10}$) | -202.33 | -0.29 | -0.17 |
| **CEC ($1,000)** | 9.51 | 24.28 | 26.18 |

We can see a substantial improvement in CEC level when the Age Pension is incorporated. For example, CEC in Case 2 increases by 155% compared to that in Case 1. This confirms that the Age Pension provided by the Australian government has been improving the general welfare of

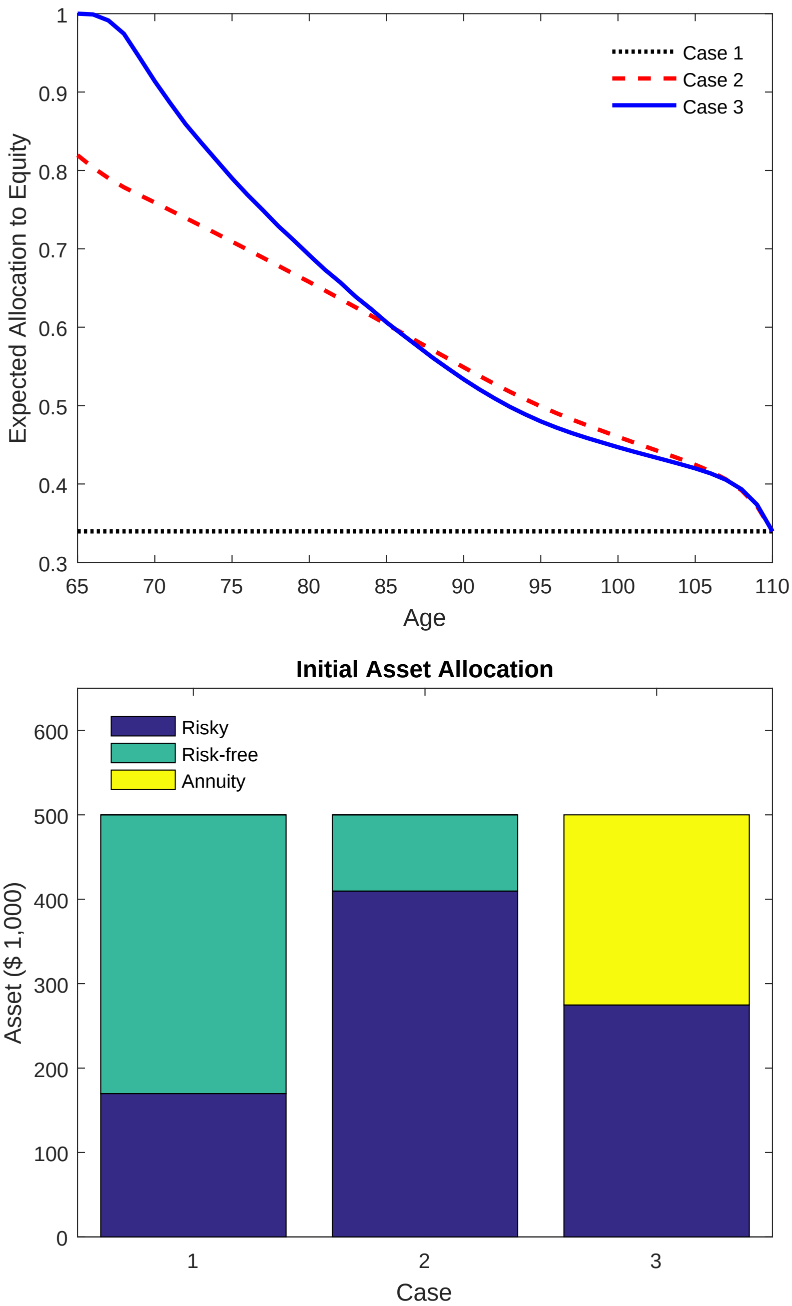


Figure 5.10: Comparison of expected allocations to equity for different cases. The top panel shows the expected allocation to risky asset as a proportion of non-annuitised wealth. The bottom panel shows the dollar amount allocation to risky asset, risk-free asset and life annuity at retirement. In Case 1, retirees have no access to the Age Pension or to life annuities; In Case 2, retirees have access to the Age Pension; In Case 3, retirees have access to the Age Pension as well as life annuities.

Table 5.4: Welfare gains of having access to the Age Pension (AP) and/or life annuities (LA). Initial wealth is $500k.

| | **AP vs. Nil** | **AP + LA vs. AP** | **AP + LA vs. Nil** |
|---|---|---|---|
| **WG ($1,000)** | 775.75 | 105.55 | 875.64 |
| **EAR (%)** | 6.36 | 1.13 | 7.04 |

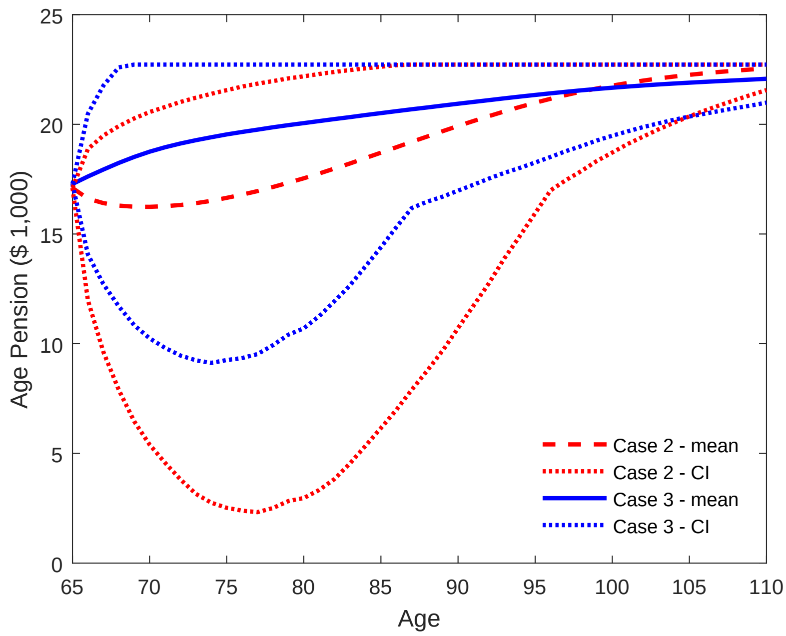


Figure 5.11: Comparison of the Age Pension entitlements for different cases. In Case 2, retirees have access to the Age Pension only. In Case 3, retirees have access to the Age Pension as well as life annuities.

the retired population to a large extent.

The wealth gap of Case 2 vs. Case 1 (i.e. the Age Pension vs. Nil) is about $776,000, reflecting the substantial welfare gains of having access to the Age Pension. In other words, the Age Pension provided by the Australian government provides the same welfare gains as if providing a lump sum subsidy of $776,000 at retirement to retirees who have half a million dollars in their account balance. On top of having entitlements to the Age Pension, the provision and efficient use of life annuities generates an extra wealth gap, which is calculated to be about $106,000. We still see significant wealth improvement by providing life annuities due to the protection of longevity risk, though this amount is relatively small compared to that of the Age Pension vs. Nil. Key reasons for the relatively small welfare gains from life annuities are (1) the Age Pension is funded via tax whereas life annuities cost a large lump sum at retirement; (2) longevity risk is largely reduced by the Age Pension, leaving smaller room for life annuities to take a role.

Comparing EAR in Case 3 and Case 1, we can conclude that the pension fund needs to grow at an extra rate of 7.04% p.a. in order to meet the welfare generated from the Age Pension and life annuities. This extra annual return is indeed very substantial in the current financial environment.

## 5.6 Results Comparison with "Technical Paper No.2: Static Models"

"Technical Paper No.2" focuses on comparing the retirement outcomes of four different static strategies. It ranks these four strategies but does not deal with finding an optimal strategy. Compared with "Technical Paper No.2", this paper ("Technical Paper No.3") has the following key differences:

- Dynamic vs. static strategies - "Technical Paper No.2" provides four static models whereas this paper focuses on finding optimal dynamic strategies. Static strategies are formulated rules that are not updated even when actual situations change over time. Dynamic strategies account for changes in market conditions and an individual's financial situation.

- Simultaneous decisions of consumption and asset allocations - It is assumed in "Technical Paper No.2" that the allocation of non-annuitised assets between risky and risk-free assets is predetermined, so the different strategies are all about consumption. By contrast, the dynamic strategies considered in "Technical Paper No.3" are simultaneous decisions of consumption and asset allocations.

- Optimal strategies - "Technical Paper No.2" ranks four static strategies based on our MDUF v1, but does not deal with finding the optimal strategy. This paper seeks optimal strategies among a wide range of different dynamic strategies, including dynamic consumption patterns, dynamic asset allocations, and annuitisation decision at retirement.

- Computational complexity - Finding optimal dynamic strategies, especially in a multi-dimensional framework as in this paper, requires backward induction based on multi-dimensional grid points. This requires high levels of computational space and time. We perform the algorithm in MATLAB. Applications of MDUF v1 to static strategies comparison are very straightforward and require very basic calculations that can be done in Excel spreadsheets.

## 6 Conclusion

One important application of MDUF v1 is seeking the optimal retirement strategy. This paper provides technical detail in terms of model set up as well as numerical examples on three case studies, where we investigate the impact of the Age Pension and life annuities.

The optimal dynamic retirement strategy provided in this paper is sophisticated and flexible to be tailored to fit better with specific institutions. Using the set of parameters specified in "Technical Paper No.1: MDUF v1 Design", we can apply the MDUF v1 to many aspects, e.g. development of retirement solutions by superannuation funds, assessment of super fund performance by regulatory authorities, default retirement product design by policymakers, etc.

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